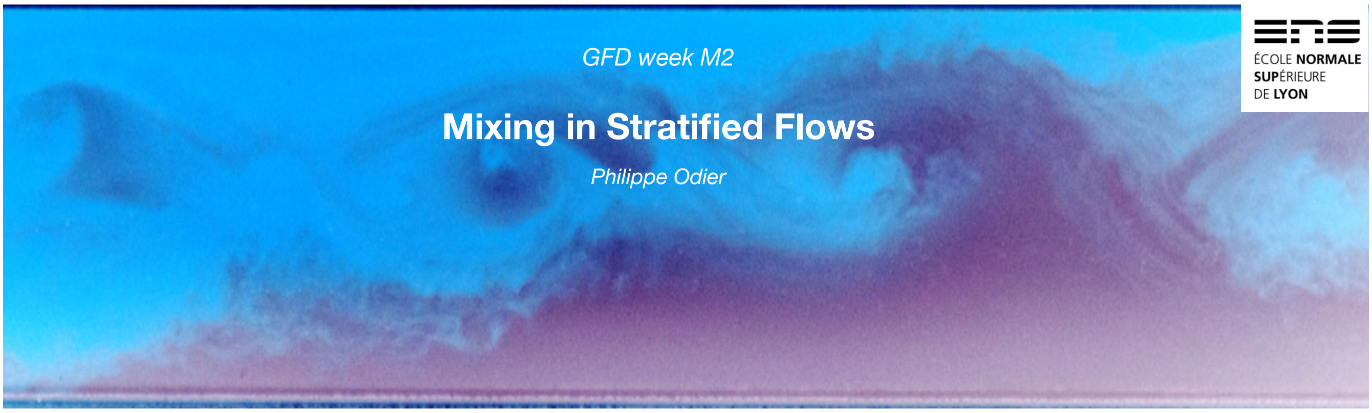


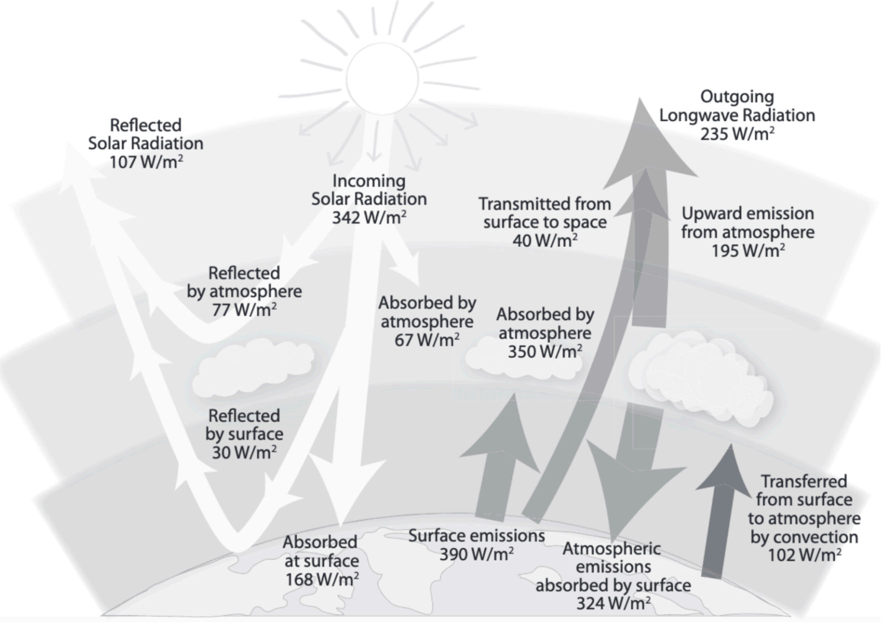
GFD week M2

Mixing in Stratified Flows

Philippe Odier

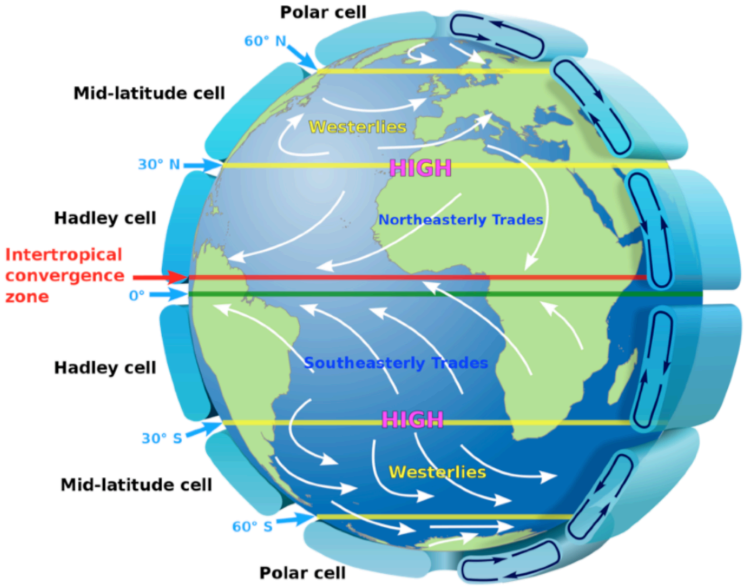
ÉCOLE **NORMALE**
SUPÉRIEURE
DE **LYON**

The atmospheric heat engine



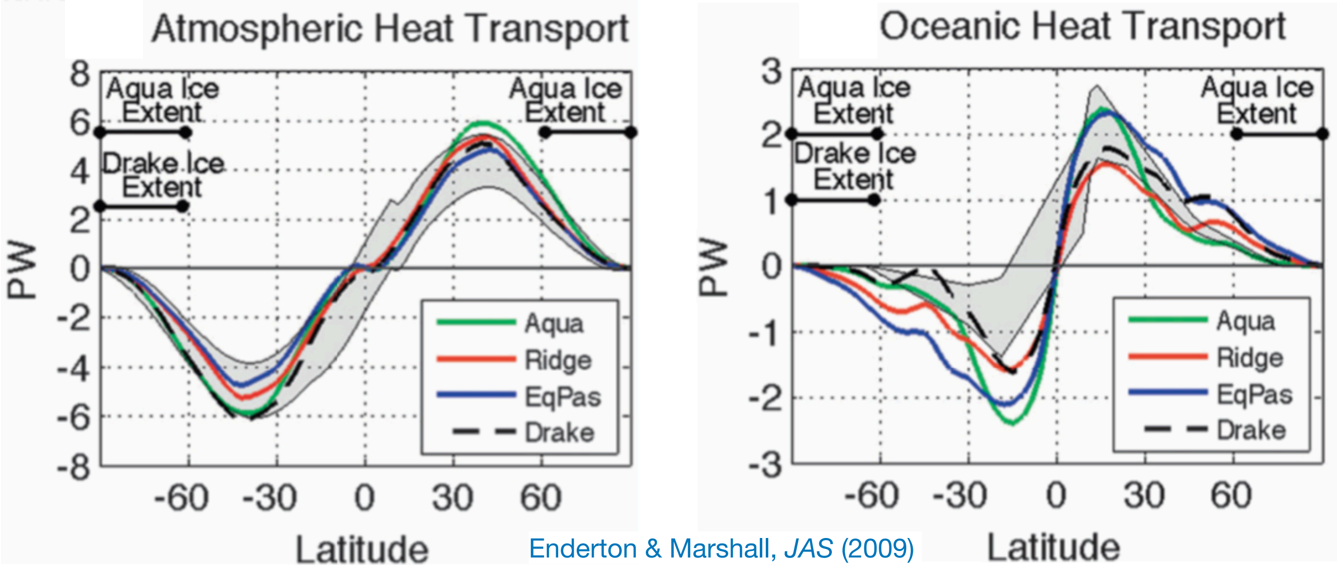
Vallis 2011, d'après Kiehl&Trenberth 1997

The atmospheric heat engine

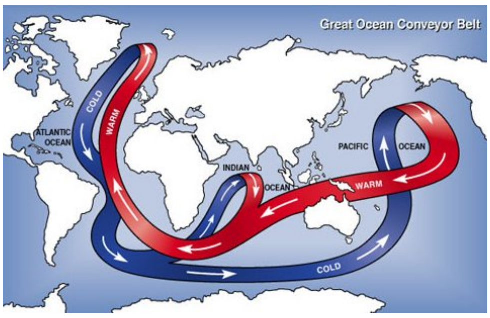


Wikipedia

The atmospheric heat engine



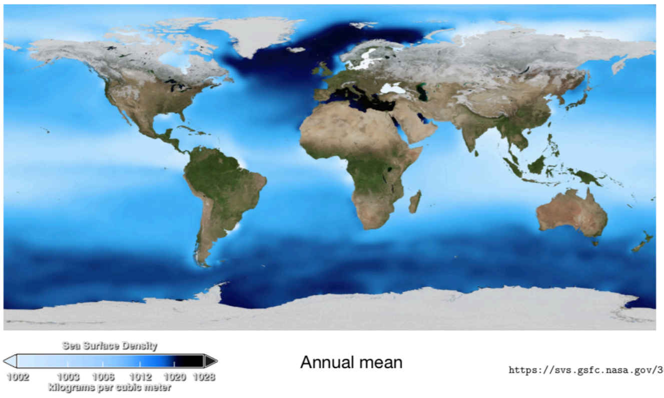
Enderton & Marshall, *JAS* (2009)



Is the ocean also
a heat engine ?

The conveyor belt

Sea Surface Density



Annual mean

https://svs.gsfc.nasa.gov/3652

https://svs.gsfc.nasa.gov/3652

Energetics of stratified turbulence

Navier-Stokes
$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla(\mathbf{u}) = -\frac{1}{\rho_0}\nabla p - \frac{\rho-\rho_0}{\rho_0} g \mathbf{e_z} + \nu \Delta \mathbf{u} \\ \partial_t u_i + u_j \partial_j u_i = -\frac{1}{\rho_0}\partial_i p - \frac{\rho-\rho_0}{\rho_0} g \delta_{iz} + \nu \partial_j \partial_j u_i \end{cases}$$

Equ. for density
$$\mathrm{D}_t \overline{\rho} = K \Delta \overline{\rho}$$

Mean-flow equ. for velocity
$$\partial_t U_i + U_j \partial_j U_i = \frac{1}{\rho_0}\partial_j(-P\delta_{ij} + \eta \partial_j U_i - \rho_0 \overline{u_i' u_j'}) - \frac{\rho-\rho_0}{\rho_0} g \delta_{iz}$$

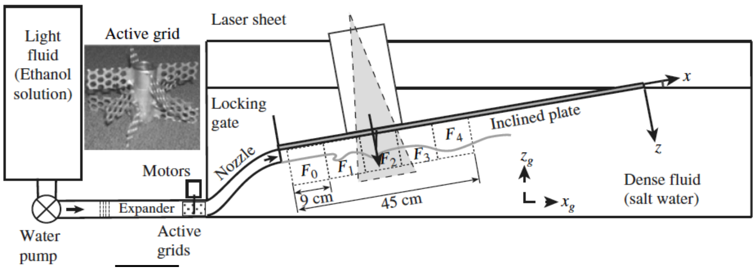
Mean-flow equ. for density
$$\mathrm{D}_t \overline{\rho} = -\partial_j(\overline{\rho' u_j'}) + K \Delta \overline{\rho} = -\nabla \cdot (\overline{\rho' \mathbf{u}'}) + K \Delta \overline{\rho}$$

Energy of the mean-flow
$$\mathrm{D}_t(\tfrac{1}{2}U_i^2) = \partial_j(-\tfrac{1}{\rho_0}U_j P + 2\nu U_i S_{ij} - U_i \overline{u_i' u_j'}) - 2\nu S_{ij} S_{ij} + \overline{u_i' u_j'} S_{ij} - \tfrac{\rho-\rho_0}{\rho_0} g W$$

Turbulent Kinetic Energy
(TKE)
$$\mathrm{D}_t(\tfrac{1}{2}\overline{u_i'^2}) = \partial_j(-\tfrac{1}{\rho_0}\overline{u_j' p'} + 2\nu \overline{u_i' s_{ij}} - \tfrac{1}{2}\overline{u_i' u_i' u_j'}) - \nu \overline{s_{ij} s_{ij}} - \overline{u_i' u_j'} S_{ij} - \tfrac{g}{\rho_0}\overline{\rho' w'}$$

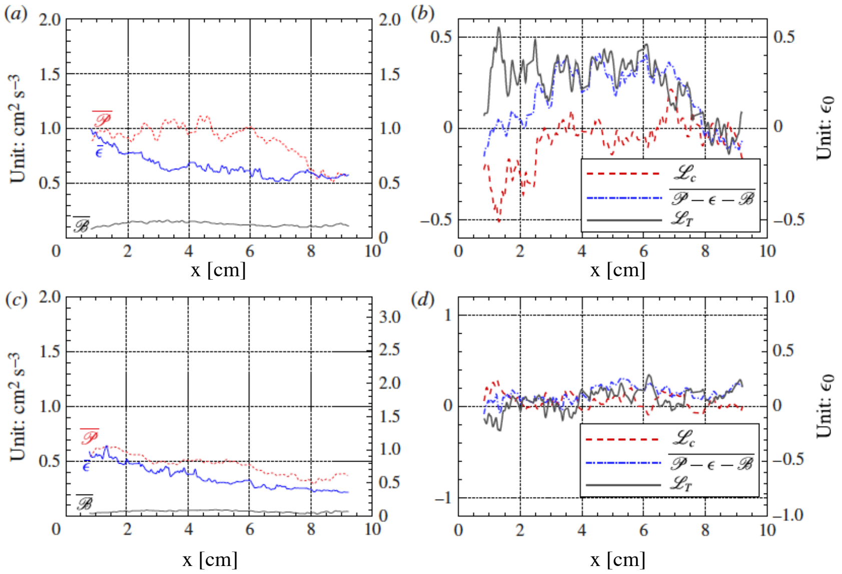
With $S_{ij} = \frac{1}{2}(\partial_j U_i + \partial_i U_j)$ $s_{ij} = \frac{1}{2}(\partial_j u_i' + \partial_i u_j')$

Energetics of stratified turbulence



Odier, Chen, Ecke,
PRL (2009)
Physics D (2012)
JFM (2014)
JFM (2017)

$$D_t\left(\tfrac{1}{2}\overline{u_i'^2}\right) = \partial_j\left(-\frac{1}{\rho_0}\overline{u_j' P} + 2\nu\overline{u_i' s_{ij}} - \tfrac{1}{2}\overline{u_i' u_i' u_j'}\right) + \varepsilon + P + B$$



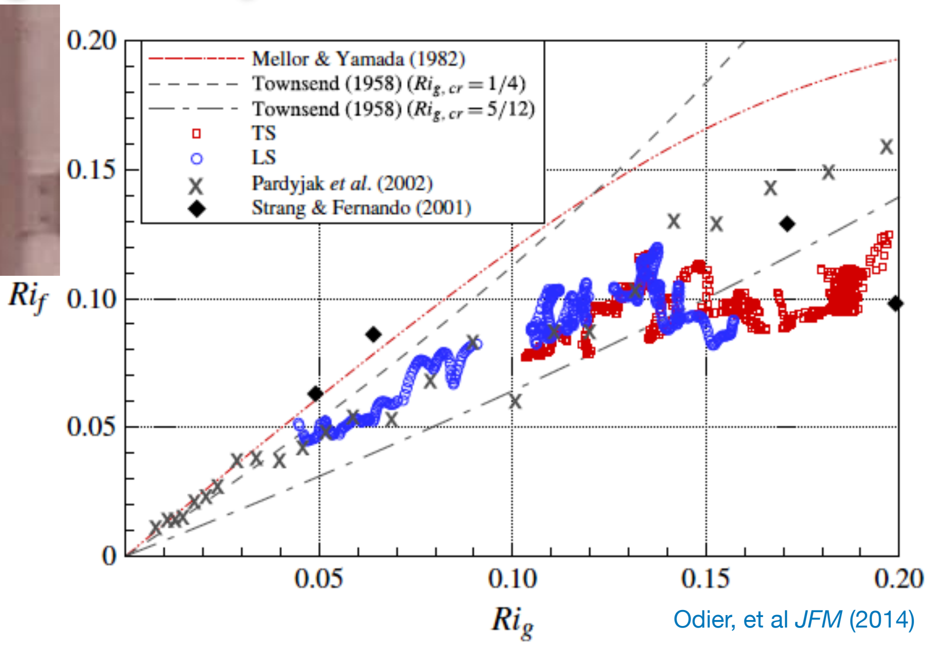
$$\varepsilon \equiv -\frac{\nu}{2}\overline{(\partial_j u_i' + \partial_i u_j')^2}$$

$$P \equiv -\overline{u_i u_j}\,\frac{\partial_j U_i + \partial_i U_j}{2}$$

$$B \equiv -\frac{g}{\rho_0}\overline{\rho' w'}$$

Mixing efficiency: local estimate

Lewis Fry Richardson
(english mathematician, physicist, meteorologist, psychologist)



Odier, et al *JFM* (2014)

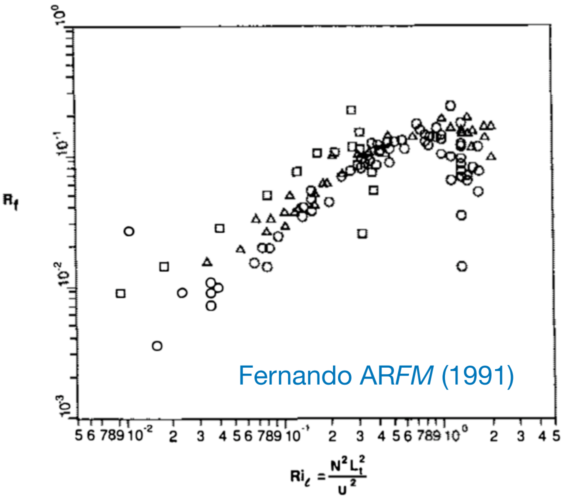
$$Fr = \frac{1}{\sqrt{Ri}}$$

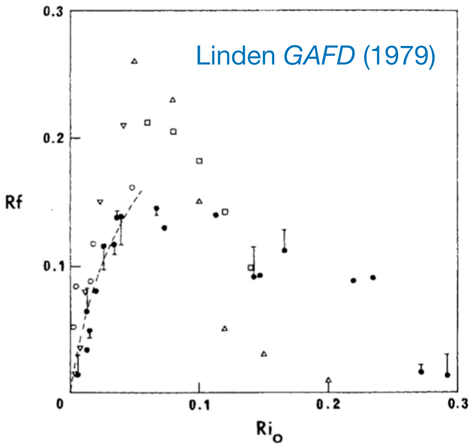
William Froude
(english engineer, hydrodynamicist and naval architect)

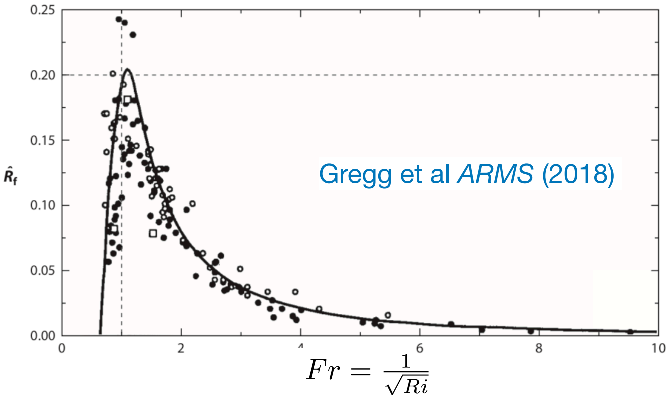
$$Ri_g \equiv \frac{N^2}{S^2} = \frac{-\frac{g}{\rho_0}\partial_z\overline{\rho}}{(\partial_z U)^2}$$

$$Ri_f \equiv \frac{B}{P} = \frac{B}{B+\varepsilon}$$

Mixing efficiency: local estimate

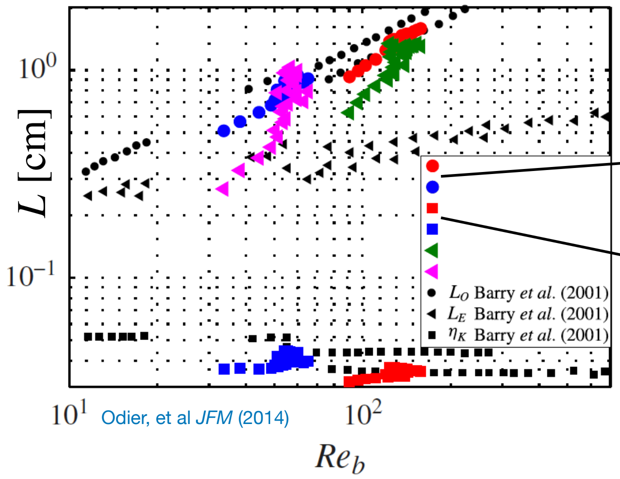

Fernando *ARFM* (1991)


Linden *GAFD* (1979)


Gregg et al *ARMS* (2018)

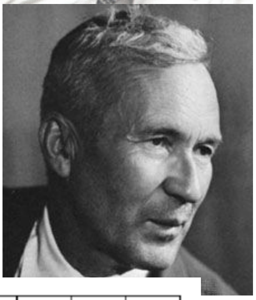
Mixing scales

Ozmidov Rostislav
Vsevolodovich
(russian oceanographer)


Odier, et al *JFM* (2014)

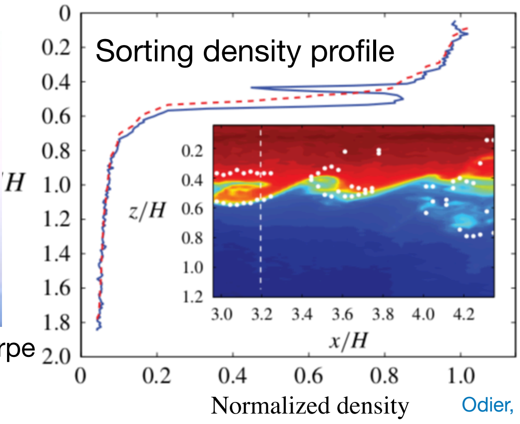
$$L_O = \left(\frac{\varepsilon}{N^3}\right)^{1/2}$$

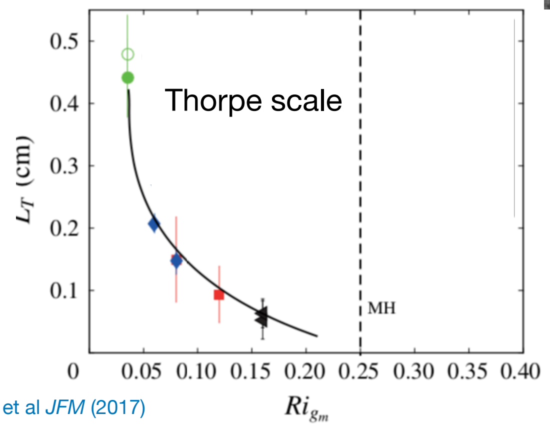
$$L_K = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$



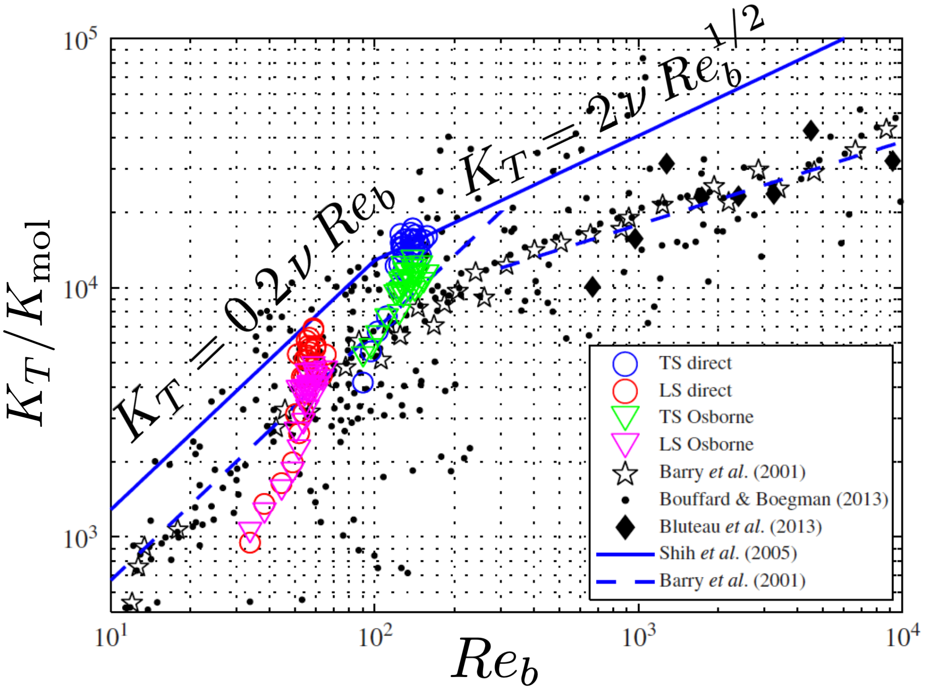
Andrey Nikolayevich
Kolmogorov
(russian mathematician)

Stephen Austen Thorpe
(british oceanographer)



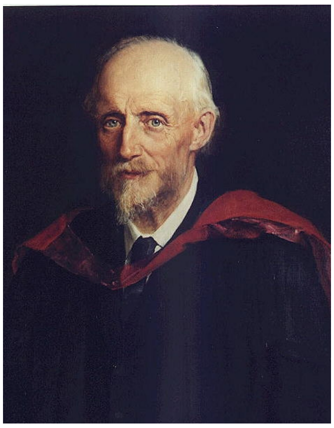

Odier, et al *JFM* (2017)

Turbulent diffusion



$$K_T \equiv \frac{\overline{\rho' w'}}{\partial_z \bar{\rho}}$$

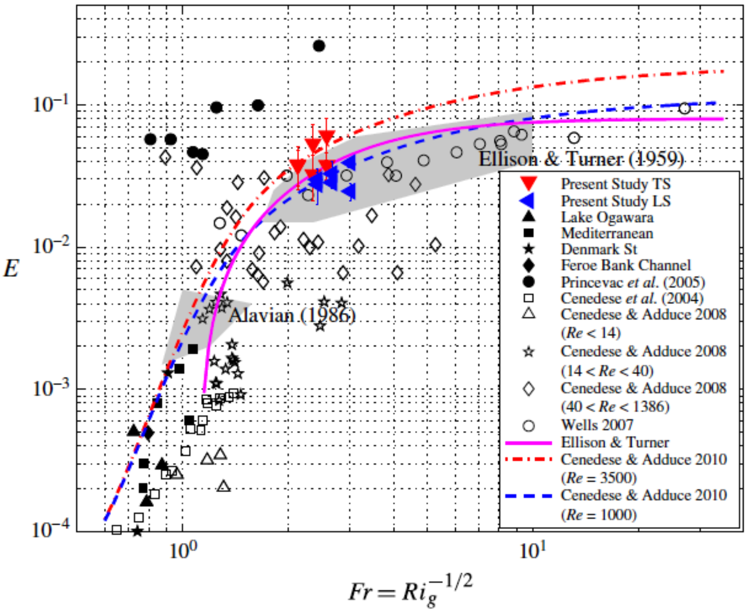
$$Re_b \equiv \frac{\varepsilon}{\nu N^2}$$



Odier, et al *JFM* (2014)

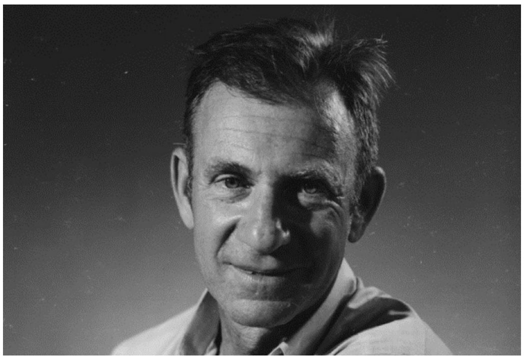
Osborne Reynolds
(irish-born professor of engineering)

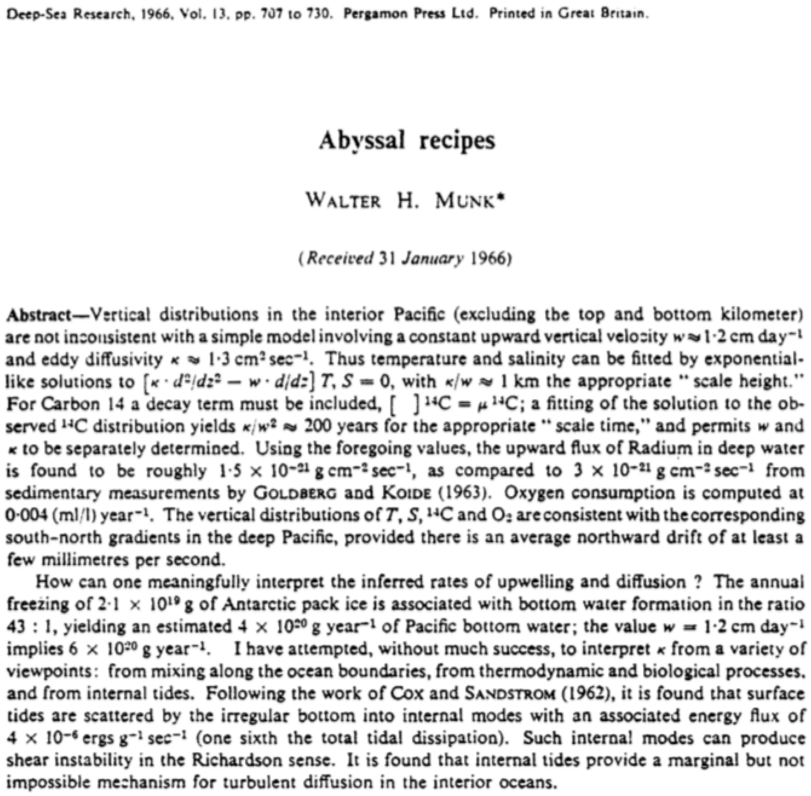
Shear flows: entrainement



Odier, et al *JFM* (2014)

Walter Munk: “Abyssal recipes”





Deep-Sea Research, 1966, Vol. 13, pp. 707 to 730. Pergamon Press Ltd. Printed in Great Britain.

Abyssal recipes

WALTER H. MUNK*

(*Received* 31 *January* 1966)

Abstract—Vertical distributions in the interior Pacific (excluding the top and bottom kilometer) are not inconsistent with a simple model involving a constant upward vertical velocity $w \approx 1{\cdot}2$ cm day^{-1} and eddy diffusivity $\kappa \approx 1{\cdot}3$ cm^2 sec^{-1}. Thus temperature and salinity can be fitted by exponential-like solutions to $[\kappa \cdot d^2/dz^2 - w \cdot d/dz]\, T, S = 0$, with $\kappa/w \approx 1$ km the appropriate “scale height.” For Carbon 14 a decay term must be included, $[\quad]\, {}^{14}\mathrm{C} = \mu\, {}^{14}\mathrm{C}$; a fitting of the solution to the observed ^{14}C distribution yields $\kappa/w^2 \approx 200$ years for the appropriate “scale time,” and permits w and κ to be separately determined. Using the foregoing values, the upward flux of Radium in deep water is found to be roughly $1{\cdot}5 \times 10^{-21}$ g cm^{-2} sec^{-1}, as compared to 3×10^{-21} g cm^{-2} sec^{-1} from sedimentary measurements by GOLDBERG and KOIDE (1963). Oxygen consumption is computed at 0·004 (ml/l) year^{-1}. The vertical distributions of T, S, ^{14}C and O_2 are consistent with the corresponding south–north gradients in the deep Pacific, provided there is an average northward drift of at least a few millimetres per second.

How can one meaningfully interpret the inferred rates of upwelling and diffusion ? The annual freezing of $2{\cdot}1 \times 10^{19}$ g of Antarctic pack ice is associated with bottom water formation in the ratio 43 : 1, yielding an estimated 4×10^{20} g year^{-1} of Pacific bottom water; the value $w = 1{\cdot}2$ cm day^{-1} implies 6×10^{20} g year^{-1}. I have attempted, without much success, to interpret κ from a variety of viewpoints: from mixing along the ocean boundaries, from thermodynamic and biological processes, and from internal tides. Following the work of Cox and SANDSTROM (1962), it is found that surface tides are scattered by the irregular bottom into internal modes with an associated energy flux of 4×10^{-6} ergs g^{-1} sec^{-1} (one sixth the total tidal dissipation). Such internal modes can produce shear instability in the Richardson sense. It is found that internal tides provide a marginal but not impossible mechanism for turbulent diffusion in the interior oceans.

Munk *DSR* (1966)

Fitting T and S advection/diffusion equation to ocean measurements

$$K_T \partial_z \partial_z (T,S) - W \partial_z (T,S) = 0$$

$$\frac{(T,S)-(T,S)_{\text{bottom}}}{(T,S)_{\text{top}}-(T,S)_{\text{bottom}}} = \frac{\exp(WH\xi/K_T)-1}{\exp(WH/K_T)-1}$$

$$H = z_{\text{top}} - z_{\text{bottom}}$$

$$\xi = \frac{z-z_{\text{bottom}}}{z_{\text{top}}-z_{\text{bottom}}}$$

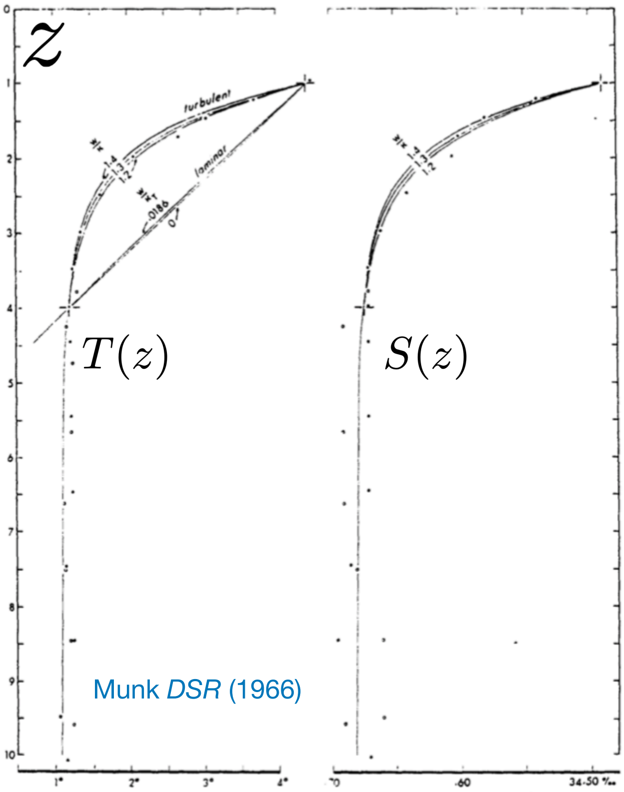


Fig. 1. Potential temperature and salinity as functions of depth (km) at stations *Snellius* 1930 : # 262, 9° 41′N, 126° 51′E, (closed circles) and *Galathea* 1951 : # 433, 9° 51′N, 126° 51′E, (open circles). Curves labeled w/κ (in units km^{-1}) are based on equations (1) and (2) for turbulent and laminar diffusion, respectively.

Munk *DSR* (1966)

Fitting radioactive elements advection/diffusion equation to ocean measurements

$$K_T \partial_z \partial_z x_{C_{14}} - W \partial_z x_{C_{14}} = \mu_c x_{C_{14}}$$

$$\begin{aligned} x_{C_{14}} = {} & C^+ \exp\left(\frac{WH}{K_T}\left(1+\frac{4K_T\mu_c}{W^2}\right)\xi\right) \\ & + C^- \exp\left(\frac{WH}{K_T}\left(1-\frac{4K_T\mu_c}{W^2}\right)\xi\right) \end{aligned}$$

$$H = z_{\text{top}} - z_{\text{bottom}}$$

$$\xi = \frac{z - z_{\text{bottom}}}{z_{\text{top}} - z_{\text{bottom}}}$$

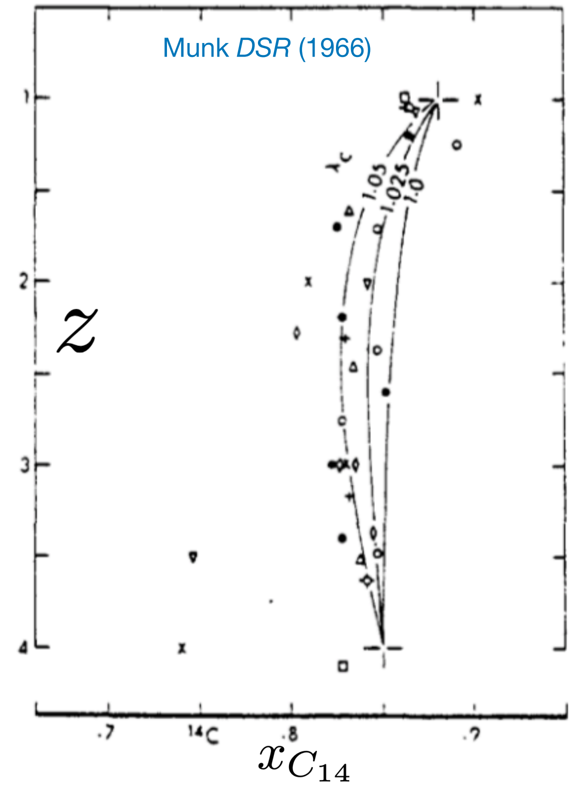
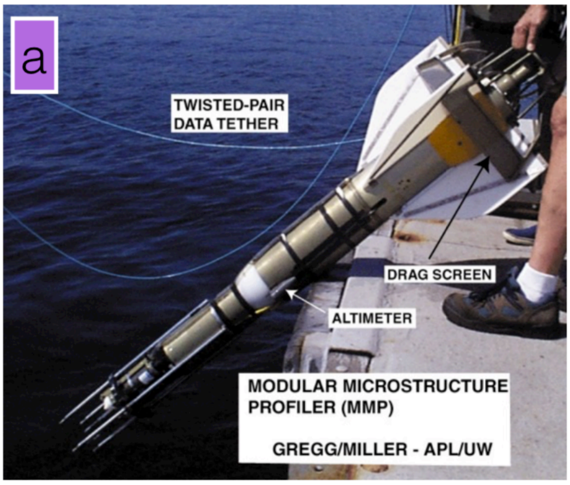

Munk *DSR* (1966)

Fig. 5. ^{14}C (relative units) as function of depth (km) (from Bien, Rakestraw and Suess, 1965). Symbols represent various Pacific stations. Curves are based on equation (6) for $\gamma = 3{\cdot}3$, and for indicated values of λ_C.

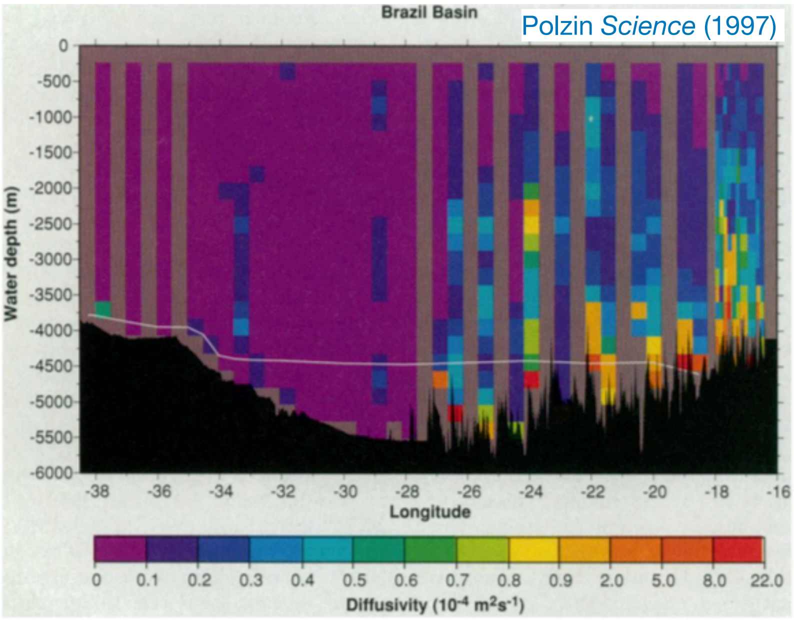
Microstructure profiler



Measured diffusivities: what is going on?

Charles “Chip” Cox

Measured diffusivities: what is going on?



Polzin *Science* (1997)

Until the processes giving rise to diffusion and advection are understood, the resulting differential equations governing the interior distribution, and their solutions, must remain what they have been for so long : a set of recipes.

Concluding sentence of Munk *DSR* (1966)

Textbooks, seminal and review articles

[1] C.-c. P. Caulfield. Open questions in turbulent stratified mixing: Do we even know what we do not know? *Phys. Rev. Fluids*, vol. 5 (2020), no. 11.

[2] W. Munk and C. Wunsch. Abyssal recipes ii: energetics of tidal and wind mixing. *Deep Sea Research Part I: Oceanographic Research Papers*, vol. 45 (1998), no. 12, pp. 1977 – 2010.

[3] W. H. Munk. Abyssal recipes. *Deep Sea Research and Oceanographic Abstracts*, vol. 13 (1966), no. 4, pp. 707–730.

[4] T. Osborn. Estimates of the local-rate of vertical diffusion from dissipation measurements. *J. Phys. Oceanogr.*, vol. 10 (1980), no. 1, pp. 83–89.

[5] S. Pope. *Turbulent Flows* (Cambridge University Press, UK, 2000).

[6] L. Shih, J. Koseff, G. Ivey, and J. Ferziger. Parameterization of turbulent fluxes and scales using homogeneous sheared stably stratified turbulence simulations. *J. Fluid Mech.*, vol. 525 (2005), pp. 193–214.

[7] H. Tennekes and J. Lumley. *A first course in turbulence* (MIT Press, 1972).

[8] G. K. Vallis. *Atmospheric and Oceanic Fluid dynamics* (Cambridge University Press, Cambridge, 2017).

[9] C. Wunsch and R. Ferrari. Vertical mixing, energy and the general circulation of the oceans. *Annu. Rev. Fluid Mech.*, vol. 36 (2004), pp. 281–314.

Suggested additional reading

[1] C. Caulfield. Layering, instabilities, and mixing in turbulent stratified flows. *Annual Review of Fluid Mechanics*, vol. 53 (2021), no. 1.

[2] R. Ferrari and C. Wunsch. Ocean circulation kinetic energy: Reservoirs, sources, and sinks. *Annual Reviews in Fluid Mechanics*, vol. 41 (2009), pp. 253–282.

[3] G. N. Ivey, K. B. Winters, and J. R. Koseff. Density stratification, turbulence, but how much mixing? *Annu. Rev. Fluid Mech.*, vol. 40 (2008), pp. 169–184.

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Origin of Illustrations

[1] T. Dauxois, S. Joubaud, P. Odier, and A. Venaille. Instabilities of Internal Gravity Wave Beams. *Ann. Rev. of Fluid Mech.*, vol. 50 (2018), pp. 131–156.

[2] Y. Dossmann, et al. Mixing by internal waves quantified using combined PIV/PLIF technique. *Exp. Fluids*, vol. 57 (2016), no. 8.

[3] Y. Dossmann, F. Pollet, P. Odier, and T. Dauxois. Mixing and formation of layers by internal wave forcing. *Journal of Geophysical Research: Oceans*, vol. 122 (2017), no. 12, pp. 9906–9917.

[4] D. Enderton and J. Marshall. Explorations of atmosphere–ocean–ice climates on an aquaplanet and their meridional energy transports. *Journal of the Atmospheric Sciences*, vol. 66 (2009), no. 6, pp. 1593 – 1611.

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[7] https://svs.gsfc.nasa.gov/3652.

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[11] P. Odier, J. Chen, and R. E. Ecke. Entrainment and mixing in a laboratory model of oceanic overflow. *J. Fluid Mech.*, vol. 746 (2014), pp. 498–535.

[12] P. Odier and R. E. Ecke. Stability, intermittency and universal Thorpe length distribution in a laboratory turbulent stratified shear flow. *J. Fluid Mech.*, vol. 815 (2017), pp. 243–256.

[13] K. Polzin, J. Toole, J. Ledwell, and R. Schmitt. Spatial variability of turbulent mixing in the abyssal ocean. *Science*, vol. 276 (1997), no. 5309, pp. 93–96.